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Geometric Dequantization and the Correspondence Problem

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On the way to settle a conjecture proposed by Mackey, we first present in detail a complete solution to the correspondence problem for systems whose configuration space is R^n. We then indicate how this can be considered as a first step in the elaboration of a geometric dequantization program which would extend the results to more general manifolds.

1. INTRODUCTION

The observables of a *classical mechanical system* (Arnold, 1978; Abraham & Marsden, 1978) on a smooth manifold M (e.g., R^n) are usually identified with the elements $\tilde{f}, \tilde{g}, \ldots$ of an algebra $\tilde{\mathfrak{A}}$ of smooth functions from the cotangent bundle of M, $T*M$ (e.g., R^{2n}) to R. $\tilde{\mathfrak{A}}$ is equipped, in addition to its usual vector space structure on R, with two algebraic structures: a Jordan product, given by pointwise multiplication; and a Lie product, namely, the Poisson bracket $\{\ ,\ \}$, e.g.,

$$\{\tilde{f}, \tilde{g}\} = \sum_{k=1}^{n} \left(\partial_{q_k}\tilde{f}\cdot\partial_{p_k}\tilde{g} - \partial_{q_k}\tilde{g}\cdot\partial_{p_k}\tilde{f}\right)$$

associated to the symplectic form ω, e.g.,

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k$$

canonically defined on $T*M$.

In contrast, the observables of a *quantum mechanical system* (Dirac, 1930; von Neumann, 1932; Mackey, 1963a) on M are usually identified with the elements $\hat{f}, \hat{g}, \ldots$ of an irreducible algebra $\hat{\mathfrak{A}}$ of linear, self-adjoint operators acting in the Hilbert space $\mathfrak{H} = L^2_C(M)$. Again, $\hat{\mathfrak{A}}$ is equipped, in addition to its usual vector space structure on R, with two algebraic structures: a Jordan product

$$\hat{f} \circ \hat{g} = (\hat{f} \cdot \hat{g} + \hat{g} \cdot \hat{f})/2 = \left[(\hat{f} + \hat{g})^2 - \hat{f}^2 - \hat{g}^2\right]/2$$

and a Lie product

$$\{\hat{f}, \hat{g}\} = [\hat{f}, \hat{g}]/i = (\hat{f} \cdot \hat{g} - \hat{g} \cdot \hat{f})/i$$

where $\hat{f} \cdot \hat{g}$ denotes the usual composition of the operators $\hat{f}$ and $\hat{g}$.

The *correspondence problem* can be stated as follows: To establish from first principles a correspondence between classical observables $\tilde{f}, \tilde{g}, \ldots$ in $\tilde{\mathfrak{A}}$, and quantum observables $\hat{f}, \hat{g}, \ldots$ in $\hat{\mathfrak{A}}$. At first sight, it could appear reasonable to expect that this correspondence (if it exists at all) would satisfy the following requirements:

(i) $a\tilde{f} + b\tilde{g} \leftrightarrow a\hat{f} + b\hat{g} \qquad \forall a, b \in R$;

(ii) $\{\hat{f} \mid \tilde{f} \in \tilde{\mathfrak{A}}\}$ generates $\hat{\mathfrak{A}}$, and in particular that this collection of operators be irreducible;

(iii) $\{\tilde{f}, \tilde{g}\} \leftrightarrow \{\hat{f}, \hat{g}\} = [\hat{f}, \hat{g}]/i$;

(iv) $\tilde{1} \leftrightarrow \hat{I}$;

to which one might wish to add some requirement to the effect that this correspondence respect at least some of the Jordan algebraic structure of $\tilde{\mathfrak{A}}$ and $\hat{\mathfrak{A}}$, e.g.,

(v) $$\tilde{f}^2 \leftrightarrow \hat{f}^2$$

The impossibility of satisfying simultaneously all these requirements is the object of well-known "*no-go theorems*"; even forgetting about (v), and relaxing (ii) to the weaker condition that $\hat{\mathfrak{A}} = \{\hat{f} \mid \tilde{f} \in \tilde{\mathfrak{A}}\}$ be of finite multiplicity would not help (Groenewold, 1946; van Hove, 1951; Chernoff, 1969; 1981; Abraham & Marsden, 1978). These negative results are in good measure responsible for the continuing interest in the correspondence problem. In particular, one has to find some fundamental reason to guide the choice of how one should relax the correspondence requirements listed above.

Partial solutions have been proposed. Exploiting the symplectic structure of $T*M$, a very natural geometric construction, known as *prequantiza-*

tion, has been recognized to produce a map from $\tilde{\mathfrak{A}}$ to a collection of linear operators on $L^2_C(T*M)$ which satisfies conditions (i), (iii), and (iv). Even in the flat case ($M = R^n$), however, the image of $\tilde{\mathfrak{A}}$ under this map consists only of first-order differential operators, and this map can therefore not preserve squares; moreover its image is not irreducible (and is, in fact, of infinite multiplicity). Further elaborations, known under the generic name of *geometric quantization*, have for essential purpose to solve the correspondence problem by extracting, in as natural a way as seems possible, an irreducible quantum theory from the image of the prequantization map. The concept of polarization plays a central role in that enterprise, and its connection with the Tomita–Takesaki theory of modular algebras and the choice of a maximal Abelian subalgebra has recently been pointed out by the author (Emch, 1981, where the basic references to the main stream of this approach are also listed). The geometric quantization scheme, which reproduces the usual quantum mechanics when $M = R^n$, seems nevertheless to meet with some difficulties in the general case. Moreover, although it does give a geometrical meaning to Dirac's warning that certain observables are more fundamental than others, it does not involve explicitly enough the following two physical elements which should be expected to play some essential role in a complete solution of the problem.

Firstly, we should notice that in the sketch of classical and quantum mechanics given in the beginning of this section no mention is made of the states of the systems considered. This oversight should be attended to, and it will be, since in the last analysis the states provide us with the expectation values, which are the primary measurable quantities attached to these theories.

Secondly, in the above formulation of quantum mechanics a universal constant, the Planck quantum of action $\hbar$, is hidden as it takes the value $\hbar = 1$ when the appropriate units are chosen. Its role in the theory has aptly been compared to that played in (special) relativity by c, the velocity of light, which can also be chosen to take the value $c = 1$ by an appropriate choice of units. It is equally well known that the prerelativistic physics can be recovered from relativistic physics by letting c tend to infinity; in particular, the Lorentz group (for $c < \infty$) contracts, in the limit $c \to \infty$, to the Galilei group. Similarly, one does expect (see, e.g., Mackey, 1963a, pp. 103–104) that the prequantum physics should be recoverable from quantum physics by taking, under mathematically controlled circumstances, the limit $\hbar \to 0$.

To incorporate these elements (i.e., the geometry, the statistical interpretation, and the classical limit) is the principal aim of the *geometric dequantization program* outlined in this paper where quantum theory is taken to be the fundamental theory from which classical mechanics—with both its Poisson bracket and its Jordan structures—is derived. As a side result, the program establishes *from first principles* a fundamental justification for the Wigner–Moyal correspondence principle which had originally (Wigner, 1932; Moyal, 1949) been proposed on an admittedly *ad hoc* basis.

2. PRE-PHASE-SPACE QUANTUM MECHANICS

With the primary purpose of first presenting the program in as simple terms as possible, Sections 2–4 deal specifically with the case $M = R^n$, and we explain in Section 5 how one should proceed to lift this restriction.

We take as our starting point the formulation (Mackey, 1963a) of quantum mechanics according to which the algebra of observables is obtained from the fundamentally geometric concept of an irreducible system of imprimitivity based on M ($= R^n$). Namely, with $\hbar$ fixed, say in (0, 1], $\hat{\mathfrak{A}}_\hbar$ will be generated from two families of operators, acting together irreducibly on some Hilbert space $\mathfrak{H}$: the first family is a projection-valued measure

$$Q_\hbar\colon \Delta \in \mathfrak{B}(M) \mapsto Q_\hbar(\Delta) \in \mathfrak{P}(\mathfrak{H})$$

and the second family is a continuous unitary representation

$$U_\hbar\colon a \in G \mapsto U_\hbar(a) \in \mathfrak{U}(\mathfrak{H})$$

satisfying

$$U_\hbar(a) Q_\hbar(\Delta) U_\hbar(a^{-1}) = Q_\hbar(a[\Delta]) \tag{1}$$

where $\mathfrak{B}(M)$ is the algebra of Borel subsets of M; $\mathfrak{P}(\mathfrak{H})$ is the lattice of projectors on the Hilbert space $\mathfrak{H}$; G is a Lie group, acting transitively and freely on M, and taken here ($M = R^n$) to be the group of translations $\{a\colon x \in M \mapsto x + a \in M\}$; and finally, $\mathfrak{U}(\mathfrak{H})$ denotes the group of unitary operators on $\mathfrak{H}$.

Together with the usual Cartesian coordinate system on R^n, the projection-valued measure $Q_\hbar$ defines a family $\{Q_\hbar^{(k)} | k = 1, 2, \ldots, n\}$ of mutually commuting self-adjoint operators, acting on $\mathfrak{H}$, and identified as "position" observables. The "momenta" $\{P_\hbar^{(k)} | k = 1, 2, \ldots, n\}$ canonically associated to these position observables are defined, via Stone's theorem, as the self-adjoint generators of $U_\hbar(G)$, namely,

$$U_\hbar(a) = \exp(-ia \cdot P_\hbar / \hbar) \qquad \forall a \in G \tag{2}$$

Conversely $Q_\hbar$ defines a continuous representation of $\hat{R}^n$, namely,

$$V_\hbar : \hat{a} \in \hat{R}^n \mapsto \exp(-i\hat{a}\cdot Q_\hbar/\hbar) \in \mathfrak{U}(\mathfrak{H}) \tag{3}$$

Upon defining for every $z=(a,\hat{a}) \in R^n \times \hat{R}^n$ the unitary operator

$$\begin{aligned} E_\hbar(z) &= \exp(-ia\cdot\hat{a}\hbar/2)U_\hbar(\hbar a)V_\hbar(\hbar\hat{a}) \\ &= \exp[-i(a\cdot P_\hbar + \hat{a}\cdot Q_\hbar)] \end{aligned} \tag{4}$$

one obtains an equivalent form of (1), namely,

$$E_\hbar(z_1)E_\hbar(z_2) = \chi_\hbar(z_1, z_2)E_\hbar(z_1+z_2) \tag{5a}$$

with

$$\chi_\hbar(z_1, z_2) = \exp[i\sigma(z_1, z_2)\hbar/2] \tag{5b}$$

and

$$\sigma(z_1, z_2) = a_1\cdot\hat{a}_2 - \hat{a}_1\cdot a_2 \tag{5c}$$

The symplectic form σ on the Kaelerian manifold $R^{2n} \cong R^n \times \hat{R}^n \cong C^n$ will play an essential role in the sequel.

Without loss of generality (von Neumann, 1931) we can realize (5) on $L^2_C(M)$, for instance (Hepp, 1974) by

$$[E_\hbar(z)\Psi](x) = \exp[-i\hbar^{1/2}\hat{a}\cdot(x-\hbar^{1/2}a/2)]\Psi(x-\hbar^{1/2}a) \tag{6}$$

We can then *define* (Segal, 1963; see also Lavine, 1965; and Grossmann, Loupias, & Stein, 1968) for every f in $L^1_C(R^n\times\hat{R}^n)$ the *function* $\hat{f}$ of the operators $\{P^{(k)}_\hbar = -i\hbar^{1/2}\partial_k;\ Q^{(k)}_\hbar = \hbar^{1/2}x_k \mid k=1,2,\ldots,n\}$ by

$$\hat{f} = \int dz\, f(z)\cdot E_\hbar(z) \tag{7}$$

Note that $\hat{f}$ is bounded as an operator on $L^2_C(R^n)$, with $\|\hat{f}\| \leqslant \|f\|_1$; and that the adjoint of the operator $\hat{f}$ is represented by the function f^*: $z\in R^n\times\hat{R}^n \mapsto f(-z)^* \in C$. The product $\hat{f}\cdot\hat{g}$ of the operators $\hat{f}$ and $\hat{g}$ induces on $L^1_C(R^n\times\hat{R}^n)$ the twisted convolution product (Kastler, 1965):

$$f *_\hbar g : \zeta \in R^n\times\hat{R}^n \mapsto \int dz\, f(z)g(\zeta - z)\chi_\hbar(z,\zeta) \tag{8}$$

with $\chi_\hbar(z,\zeta)$ defined in (5). Consequently, the Lie bracket $[\hat{f},\hat{g}]/i\hbar$ of the operators $\hat{f}$ and $\hat{g}$ naturally induces on $L^1_C(R^n\times\hat{R}^n)$ the Lie bracket

$$\{f,g\}_\hbar:\zeta\in R^n\times\hat{R}^n\mapsto\int dz\, f(z)g(\zeta-z)\pi_\hbar(z,\zeta) \tag{9a}$$

with

$$\pi_\hbar(z,\zeta)=[\chi_\hbar(z,\zeta)-\chi_\hbar(\zeta,z)]/i\hbar \tag{9b}$$

(compare with Moyal, 1949).

This correspondence allow us to use the algebra

$$\mathfrak{A}_\hbar=\left(L^1_C(R^n\times\hat{R}^n),*_\hbar,\{\cdot,\cdot\}_\hbar\right) \tag{10}$$

to describe completely the algebra of observables

$$\hat{\mathfrak{A}}_\hbar=\left(\{\hat{f}\,|\,f\in\mathfrak{A}_\hbar\},\cdot,[\cdot,\cdot]/i\hbar\right) \tag{11}$$

equipped with the Jordan and Lie structures it inherits from the operator product. Notice that, as a subset of $\mathfrak{B}(\mathfrak{H})$, $\hat{\mathfrak{A}}_\hbar$ is independent of $\hbar\in(0,1]$; is stable under Hermitian conjugation; is irreducible, and hence dense in $\mathfrak{B}(\mathfrak{H})$; is stable under the action of G; and contains all the quantum information introduced so far. These algebraic properties are faithfully reflected in $\mathfrak{A}_\hbar$. The advantage of using $\mathfrak{A}_\hbar$ is that it emphasizes in both of its product structures the role of the parameter $\hbar\in(0,1]$, and hence the information that an identification of the basic observables of the theory has been made (namely, that $P_\hbar$ and $Q_\hbar$ have been defined as we did) involving explicitly the physical constant $\hbar$. Notice moreover that, although the quantum theory itself is only defined for $\hbar>0$, one has

$$\begin{aligned}&\lim_{\hbar\to0}\chi_\hbar(z,\zeta)=1\\&\lim_{\hbar\to0}\pi_\hbar(z,\zeta)=\sigma(z,\zeta)\end{aligned} \tag{12}$$

thus indicating that as $\hbar$ "reaches" (in some sense which we shall make precise) the value $\hbar=0$, the algebra $\mathfrak{A}_\hbar$ becomes Abelian, while still "remembering" its quantum Lie bracket through the symplectic form σ.

The last element needed to complete our description of quantum mechanics on R^n is the notion of state. Among the several possible definitions, we choose one, and then show its relations with some others in Lemma 1 and in the examples following that general result. Specifically, a continuous function φ: $R^n \times \hat{R}^n \to C$ is said to be of *$\hbar$-positive type* if it satisfies for some $\hbar \in (0, 1]$:

$$\sum_{j,k=1}^{m} \lambda_k^* \lambda_j \varphi(z_j - z_k) \chi_\hbar(z_j, z_k) \geqslant 0$$

$$\forall m \in Z^+, \forall \{\lambda_1, \ldots, \lambda_m\} \subset C, \forall \{z_1, \ldots, z_m\} \subset R^n \times \hat{R}^n \tag{13}$$

φ is said to be a *state* on $\mathfrak{A}_\hbar$ if it satisfies moreover the normalization condition $\varphi(0) = 1$.

In the sequel, for every $\varphi \in L_C^\infty(R^n \times \hat{R}^n)$ we use again the same symbol φ to denote the map from $L_C^1(R^n \times \hat{R}^n)$ to C defined by

$$\langle \varphi; f \rangle = \int dz\, \varphi(z) f(z) \tag{14}$$

Lemma 1. (a)For a continuous function φ: $R^n \times \hat{R}^n \to C$ the following conditions are equivalent:

(i) φ is of $\hbar$-positive type;
(ii) φ is bounded, and for every continuous function f: $R^n \times \hat{R}^n \to C$ of compact support $\langle \varphi; f^* *_\hbar f \rangle \geqslant 0$.

(b) For any φ in $L_C^\infty(R^n \times \hat{R}^n)$, the following conditions are equivalent:

(i) φ: $f \in \mathfrak{A}_\hbar \mapsto \langle \varphi; f \rangle \in C$ is positive;
(ii) φ is locally almost everywhere equal to a continuous function of $\hbar$-positive type.

Proof. The condition that a continuous function be of $\hbar$-positive type is the generalization (for $\hbar > 0$) of the classical condition (obtained at $\hbar = 0$) that a continuous function be of positive type. In the latter case the above lemma is a well-known result (see, e.g., Dixmier, 1969, Section XIII), and one could extend step by step the classical proofs to the present case. It is, however, interesting for our purpose in this paper to notice that this is more than a mere analogy: the present case can be made to be contained as a particular case of the classical theory for locally compact, not necessarily Abelian, groups. Since (5) is a projective representation of $R^n \times \hat{R}^n$, we know (Bargmann, 1954) that it can be seen as a unitary representation of an extension $\Xi_\hbar$ of $R^n \times \hat{R}^n$ by the multiplicative group Ω of the complex numbers of modulus 1. Let indeed $\Xi_\hbar$ denote the Cartesian product

$$\Xi_\hbar = \Omega \times R^n \times \hat{R}^n \tag{15}$$

equipped with the composition law:

$$(\omega_1, z_1)(\omega_2, z_2) = (\omega_1\omega_2\chi_\hbar(z_1, z_2), z_1 + z_2) \tag{16}$$

Clearly the continuous group homomorphisms

$$\pi\colon (\omega, z) \in \Xi_\hbar \mapsto z \in R^n \times \hat{R}^n$$

$$i\colon \quad \omega \in \Omega \mapsto (\omega, 0) \in \Xi_\hbar$$

satisfy the exact sequence condition

$$1 \to \Omega \xrightarrow{i} \Xi_\hbar \xrightarrow{\pi} R^n \times \hat{R}^n \to 0 \tag{17}$$

which exhibits explicitly the group extension structure of $\Xi_\hbar$. For any φ: $R^n \times \hat{R}^n \to C$, continuous or in L_C^∞, we can now define

$$\Phi_\hbar\colon (\omega, z) \in \Xi_\hbar \mapsto \omega\varphi(z) \in C \tag{18}$$

and check immediately that $\Phi_\hbar$ satisfies the various conditions of the lemma exactly when φ does, with, however, $\hbar$-positivity on $R^n \times \hat{R}^n$ now replaced by the classical positive-type condition on the non-Abelian, locally compact group $\Xi_\hbar$:

$$\sum_{j,k=1}^{m} \lambda_k^* \lambda_j \Phi_\hbar(\xi_k^{-1}\xi_j) \geq 0$$

$$\forall m \in Z^+, \forall\{\lambda_1, \ldots, \lambda_m\} \subset C, \qquad \forall\{\xi_1, \ldots, \xi_m\} \subset \Xi_\hbar \tag{19}$$

Hence the classical equivalences can be used directly. ■

By way of examples for states in the sense chosen here, consider any density matrix ρ on $\mathfrak{H} = L_C^2(R^n)$, i.e., any positive, trace class operator on $\mathfrak{H}$ with $\mathrm{Tr}\,\rho = 1$. Clearly

$$\varphi_\hbar\colon z \in R^n \times \hat{R}^n \mapsto \mathrm{Tr}\,\rho E_\hbar(z) \in C \tag{20}$$

is a continuous function of $\hbar$-positive type, with $\varphi_\hbar(0) = 1$, and hence is a state on $\mathfrak{A}_\hbar$. Moreover since ρ is of Hilbert–Schmidt class, $\varphi_\hbar$ belongs to L_C^2 (Segal, 1963), and conversely every L_C^2-state is obtained from a density matrix. This gives thus a precise characterization of those states on $\mathfrak{A}_\hbar$ which extend to normal states on $\mathfrak{B}(\mathfrak{H})$.

3. CLASSICAL STATES

In Section 2 above, our exposition of quantum theory was developed without any reference to a possible underlying classical theory. In the present section and the next, the classical theory will be made to emerge as a consequence of a limiting procedure exploiting systematically the notion of convergence of quantum expectation values as $\hbar \to 0$.

Definition 1. A family $\{\varphi_\hbar \mid \hbar \in (0,1]\}$ of states is said to be a classical state if it is convergent in the $\sigma(L^\infty, L^1)$ topology as $\hbar \to 0$.

Proposition 1. (a) On a family $\{\varphi_\hbar \mid \hbar \in (0,1]\}$ of states the following conditions are equivalent:

(i) $\{\varphi_\hbar \mid \hbar \in (0,1]\}$ is a classical state;

(ii) $\{\varphi_\hbar \mid \hbar \in (0,1]\}$ converges uniformly on compacta as $\hbar \to 0$.

(b) When the conditions of part (a) are realized there exists a continuous function φ_0: $R^n \times \hat{R}^n \to C$ with the following properties:

(i) $\varphi_\hbar \to \varphi_0$ uniformly on compacta as $\hbar \to 0$;

(ii) $\varphi_0(0) = 1$;

(iii) φ_0 is of positive type.

Proof. (aii) clearly implies (ai). Indeed, let φ_0 be the continuous function obtained as uniform limit on compacta from $\{\varphi_\hbar \mid \hbar \in (0,1]\}$; and f be in L^1. Since the algebra $\mathfrak{K}$ of continuous functions with compact support is dense in L^1, there exists, for each $\varepsilon > 0$, $g \in \mathfrak{K}$ such that $\|f-g\|_1 < \varepsilon/3$. For that g we have from (aii) that there exists $\delta > 0$ such that

$$|\langle \varphi_\hbar; g\rangle - \langle \varphi_0; g\rangle| \leq \varepsilon/3 \qquad \forall \hbar \in [0,\delta]$$

We have then

$$|\langle \varphi_\hbar; f\rangle - \langle \varphi_0; f\rangle| \leq |\langle \varphi_\hbar; f-g\rangle| + |\langle \varphi_\hbar - \varphi_0; g\rangle| + |\langle \varphi_0; g-f\rangle|$$

$$\leq (\|\varphi_\hbar\|_\infty + \|\varphi_0\|_\infty) \cdot \|f-g\|_1 + \varepsilon/3 \leq \varepsilon$$

since $\|\varphi_\hbar\|_\infty = 1$ for all $\hbar > 0$, and thus $\|\varphi_0\|_\infty = 1$. We have thus obtained (ai) from (aii). Conversely, let us now assume (ai). There exists then $\varphi_0 \in L^\infty$ such that $\varphi_\hbar \to \varphi_0$ in the $\sigma(L^\infty, L^1)$ topology. Let now $\varepsilon > 0$, and $f \in \mathfrak{K}$, with support K. With $*_\hbar$ denoting the convolution product of $\mathfrak{A}_\hbar$, and $*$ the ordinary convolution product of L^1, we have from (ai) that there exists $\delta_1 > 0$ such that

$$|\langle \varphi_\hbar; f^* * f\rangle - \langle \varphi_0; f^* * f\rangle| \leq \varepsilon/2 \qquad \forall \hbar \in [0,\delta_1]$$

On the other hand

$$|\langle\varphi_\hbar; f^* *_\hbar f\rangle - \langle\varphi_\hbar; f^* * f\rangle| = \left|\int dz\, \varphi_\hbar(z) \int d\zeta\, f(\zeta)^* f(z+\zeta)[\chi_\hbar(z,\zeta)-1]\right|$$

$$\leqslant \|\varphi_\hbar\|_\infty \cdot \|f\|_\infty^2 \cdot \mu(K)\cdot\mu(K')$$

$$\sup_{z\in K',\zeta\in K} |\chi_\hbar(z,\zeta)-1|$$

where K' is some compact such that $z+\zeta \in K$ whenever $\{\zeta\in K, z\in K'\}$. From the explicit form of $\chi_\hbar$ [see equation (5)], one therefore sees that there exists $\delta_2 > 0$ such that

$$|\langle\varphi_\hbar; f^* *_\hbar f\rangle - \langle\varphi_\hbar; f^* * f\rangle| \leqslant \varepsilon/2 \qquad \forall\hbar\in[0,\delta_2]$$

We have thus with $\delta = \mathrm{Min}\{\delta_1, \delta_2\}$:

$$|\langle\varphi_\hbar; f^* *_\hbar f\rangle - \langle\varphi_0; f^* * f\rangle| \leqslant \varepsilon \qquad \forall\hbar\in[0,\delta]$$

Since $\varphi_\hbar$ is of $\hbar$-positive type, it follows from the above inequality that φ_0 is of positive type. By the classical result referred to in the proof of Lemma 1, φ_0 is locally almost everywhere equal to a continuous function of positive type. For our purpose we can therefore assume w.l.g. φ_0 to be continuous of positive type. From $\|\varphi_\hbar\| = 1$ $\forall\hbar\in(0,1]$ follows $\|\varphi_0\| = 1$, and hence $\varphi_0(0) = 1$. To prove that $\varphi_\hbar \to \varphi_0$ uniformly on compacta now requires only a straightforward modification of the classical argument (e.g., Dixmier, 1969, Theorem 13.5.2). This modification runs indeed as follows. Let φ_0 be as above, and $\varepsilon > 0$. As in the classical case, we start by noticing that φ_0 continuous, and $\varphi_0(0) = 1$ imply the existence of $a > 0$ such that $|1-\varphi_0(z)| \leqslant \varepsilon$ for all z in the ball $B(a)$ of radius a. With b denoting the volume of $B(a)$, and Θ_a the indicator function of $B(a)$, consider the function $f_a = b^{-1}\Theta_a$ in L^1 and remark that $\|f_a\|_1 = 1$. We will use this function in the following majorization:

$$|\varphi_\hbar(z) - \varphi_0(z)| \leqslant \sum_{k=1}^{4} \alpha_\hbar^{(k)}(z)$$

where

$$\alpha_\hbar^{(1)}(z) = |\varphi_\hbar(z) - (\varphi_\hbar *_\hbar f_a)(z)|$$

$$\alpha_\hbar^{(2)}(z) = |(\varphi_\hbar *_\hbar f_a)(z) - (\varphi_\hbar * f_a)(z)|$$

$$\alpha_\hbar^{(3)}(z) = |(\varphi_\hbar * f_a)(z) - (\varphi_0 * f_a)(z)|$$

$$\alpha_\hbar^{(4)}(z) = |(\varphi_0 * f_a)(z) - \varphi_0(z)|$$

For $k=1$ and 4, we can use at fixed $\hbar$ the extension from $\varphi_\hbar$ on $R^n \times \hat{R}^n$ to $\Phi_\hbar$ on $\Xi_\hbar$ given in the proof of Lemma 1, and proceed similarly with f_a, to conclude directly from the classical argument that there exists $\delta'_a > 0$ such that

$$\left.\begin{array}{l}\alpha_\hbar^{(1)}(z) \leqslant 2\varepsilon^{1/2} \\ \alpha_\hbar^{(4)}(z) \leqslant 2\varepsilon^{1/2}\end{array}\right\} \qquad \forall z \in B(a), \qquad \forall \hbar \in (0, \delta'_a]$$

$\alpha_\hbar^{(3)}(z)$ involves only the ordinary convolution product in L^1 so that we can use the classical argument (Dixmier, 1969, Lemma 13.5.1) to conclude, since $\|\varphi_\hbar\| = 1 = \|\varphi_0\|$, that the $\sigma(L^\infty, L^1)$ convergence of $\varphi_\hbar$ to φ_0 implies the existence of some $\delta''_a > 0$ such that

$$\alpha_\hbar^{(3)}(z) \leqslant \varepsilon, \qquad \forall z \in B(a), \qquad \forall \hbar \in (0, \delta''_a]$$

Finally we have for all z in $B(a)$

$$\begin{aligned}\alpha_\hbar^{(2)}(z) &= b^{-1}\left|\int_{B(a)} d\zeta\, \varphi_\hbar(z-\zeta)\left[\chi_\hbar(\zeta, z) - 1\right]\right| \\ &\leqslant \sup_{\zeta, z \in B(a)} |\chi_\hbar(\zeta, z) - 1|\end{aligned}$$

We thus conclude from the explicit form of $\chi_\hbar$ in (5), that there exists some $\delta'''_a > 0$ such that

$$\alpha_\hbar^{(2)}(z) \leqslant \varepsilon, \qquad \forall z \in B(a), \qquad \forall \hbar \in (0, \delta'''_a]$$

Upon collecting these results, we obtain that there exists some $\delta_a = \mathrm{Min}\{\delta'_a, \delta''_a, \delta'''_a\}$ such that

$$|\varphi_\hbar(z) - \varphi_0(z)| \leqslant 2\varepsilon + 4\varepsilon^{1/2}, \forall z \in B(a), \forall \hbar \in (0, \delta_a]$$

From this follows that $\varphi_\hbar$ converges uniformly on compacta to φ_0, thus completing the proof of the proposition. ■

Remark. An alternate proof of this proposition could have been obtained in complete parallel with the classical one (Dixmier, 1969, Theorem 13.5.2), by using the fact that the state $\varphi_\hbar$ on $\mathfrak{A}_\hbar$ generates via the GNS construction, a representation of the canonical commutation relations (CCR $\hbar$). The proof was presented in the manner chosen here only to emphasize that this apparent generalization is not actually needed, and that the result can in fact be seen as a straightforward application (up to one additional ε coming from $\alpha_\hbar^{(2)}$) of a well-known argument.

Having obtained, through condition (aii) of the above proposition, an effective criterion to recognize classical states in a quantum theory with explicit dependence on $\hbar$, we now give a few examples.

(a) The *ground state* of the harmonic oscillator

$$H_\hbar = \left(P_\hbar^2 + \omega^2 Q_\hbar^2\right)/2 \tag{21}$$

is described in the representation (6) by the vector Φ_0 given by

$$\Phi_0(x) = (\omega/\pi)^{1/4} \exp(-\omega x^2/2) \tag{22}$$

and the corresponding state is the function

$$\begin{aligned} \varphi_{0,\hbar}(z) &= \left(E_\hbar(z)\Phi_0, \Phi_0\right) \\ &= \exp\left[-\hbar(\omega a^2 + \omega^{-1}\hat{a}^2)/4\right] \end{aligned} \tag{23}$$

Clearly $\{\varphi_{0,\hbar} | \hbar \in (0,1]\}$ converges, uniformly on compacta as $\hbar \to 0$, to the function $\varphi_0(z) = 1$; hence it is a classical state in the sense of Definition 1. Remark that the Fourier transform of φ_0 is the δ measure at the origin.

(b) The *Schrödinger coherent states*, of which example (a) above (with $\omega = 1$) is a particular case, are indexed by $\hat{z} = (\hat{p}, q) \in \hat{R}^n \times R^n$ and given by

$$\begin{aligned} \varphi_{\hat{z},\hbar} &: z = (a, \hat{a}) \in R^n \times \hat{R}^n \\ &\mapsto e_{\hat{z}}(z) \exp\left[-\hbar(a^2 + \hat{a}^2)/4\right] \in C \end{aligned} \tag{24}$$

where

$$e_{\hat{z}}(z) = \exp[-i(\hat{p}a + q\hat{a})]$$

These again are classical states in the sense of Definition 1, since $\{\varphi_{\hat{z},\hbar} | \hbar \in (0,1]\}$ converges, uniformly on compacta as $\hbar \to 0$, to the function

$$\varphi_{\hat{z}}: z \in R^n \times \hat{R}^n \mapsto e_{\hat{z}}(z) \in C$$

Remark again that the Fourier transform of $\varphi_{\hat{z}}$ is a δ measure, concentrated now at $\hat{z} = (\hat{p}, q) \in \hat{R}^n \times R^n$. The physical interpretation of this result is based on the remark that we have (heuristically, but this can be made rigorous easily)

$$\langle \varphi_{\hat{z},\hbar}; Q_\hbar \rangle = q; \qquad \langle \varphi_{\hat{z},\hbar}; P_\hbar \rangle = \hat{p}$$

$$\langle \varphi_{\hat{z},\hbar}; (Q_\hbar - \langle \varphi_{\hat{z},\hbar}; Q_\hbar \rangle I)^2 \rangle \langle \varphi_{\hat{z},\hbar}; (P_\hbar - \langle \varphi_{\hat{z},\hbar}; P_\hbar \rangle I)^2 \rangle = \hbar^2/4 \tag{25}$$

i.e., these states are centered at $(\hat{p}, q)$ with minimal quantum uncertainty. This property is evidently well known (Schrödinger, 1926), and it certainly played an important role in the formation of the physical intuition that, as $\hbar \to 0$, quantum theory "approaches" a classical limit. The structure of these states actually deserves a closer examination, which leads to the formulation of the more general examples (c) and (d) below.

We can indeed rewrite (24) in the form

$$\varphi_{\hat{z},\hbar}(z) = \langle \varphi_{0,\hbar}; E_{\hat{z},\hbar}(z) \rangle$$

with

$$E_{\hat{z},\hbar}(z) = e_{\hat{z}}(z) E_\hbar(z) \tag{26}$$

and notice (Hepp, 1974; Roepstorff, 1970) that $E_{\hat{z},\hbar}$ is again an irreducible representation of the CCR($\hbar$), so that there exists (von Neumann, 1931) a unitary operator $C_\hbar(\hat{z})$ defined, uniquely up to a complex number of modulus 1, by the relation

$$C_\hbar(\hat{z})^* E_\hbar(z) C_\hbar(\hat{z}) = e_{\hat{z}}(z) E_\hbar(z), \qquad \forall z \in R^n \times \hat{R}^n \tag{27}$$

Consequently $\varphi_{\hat{z},\hbar}$ is a vector state in the representation (6), namely

$$\varphi_{\hat{z},\hbar}(z) = (E_\hbar(z)\Phi_{\hat{z},\hbar}, \Phi_{\hat{z},\hbar})$$

with

$$\Phi_{\hat{z},\hbar} = C_\hbar(\hat{z})\Phi_0 \tag{28}$$

where Φ_0 is given by (22) (with $\omega = 1$). It has also been noticed that the convergence of $\varphi_{\hat{z},\hbar}$ to $\varphi_{\hat{z}} = e_{\hat{z}}$ is in fact a particular consequence (Hepp, 1974) of the strong operator convergence, in the representation (6), of $E_\hbar(z)$ to I as $\hbar \to 0$, which gives, through (27),

$$\operatorname*{s\text{-}lim}_{\hbar \to 0} C_\hbar(\hat{z})^* E_\hbar(z) C_\hbar(\hat{z}) = e_{\hat{z}}(z) \cdot I \tag{29}$$

This leads to a third class of examples.

(c) The *Perelomov coherent states*. For any vector $\Psi \in \mathfrak{H}$, the collection $\{\Psi_{\hat{z},\hbar} = C_\hbar(\hat{z})\Psi \mid \hat{z} \in \hat{R}^n \times R^n\}$ form, by definition, an (overcomplete) set of coherent states in the sense of (Perelomov, 1972). We now note that in our description, the states

$$\psi_{\hat{z},\hbar}(z) = (E_\hbar(z)\Psi_{\hat{z},\hbar}, \Psi_{\hat{z},\hbar}) \tag{30}$$

although they do not necessarily have, for $\hbar > 0$, the minimal quantum uncertainty compatible with the Heisenberg principle, are nevertheless concentrated more and more around $z = (\hat{p}, q)$ as $\hbar \to 0$, since as a consequence of (29) we have again

$$\lim_{\hbar \to 0} \psi_{\hat{z}, \hbar}(z) = e_{\hat{z}}(z) \tag{31}$$

(d) More *general coherent states* can still be obtained. Indeed since $C_\hbar(\hat{z})$ and $E_\hbar(z)$ are unitary, (29) holds with the ultrastrong topology substituted for the strong topology. Consequently, we have for every density matrix ρ on $\mathfrak{H}$ and every $\hat{z} \in \hat{R}^n \times R^n$

$$\lim_{\hbar \to 0} \rho_{\hat{z}, \hbar}(z) = e_{\hat{z}}(z) \tag{32a}$$

where

$$\rho_{\hat{z}, \hbar}(z) = \operatorname{Tr} \rho_{\hat{z}, \hbar} E_\hbar(z) \tag{32b}$$

and

$$\rho_{\hat{z}, \hbar} = C_\hbar(\hat{z}) \rho C_\hbar(\hat{z})^* \tag{32c}$$

(e) Finally, another collection of examples of classical states is provided by the *canonical equilibrium states* at natural temperature $\beta = 1/kT$. For instance, the canonical equilibrium states for the quantum harmonic oscillator (21) are defined by the density matrices

$$\rho_{\beta, \hbar} = \exp(-\beta H_\hbar) / \operatorname{Tr} \exp(-\beta H_\hbar) \tag{33}$$

from which one computes

$$\begin{aligned} \rho_{\beta, \hbar}(z) &= \operatorname{Tr} \rho_{\beta, \hbar} E_\hbar(z) \\ &= \exp\left[-\hbar \Theta_\hbar \cdot (\omega a^2 + \omega^{-1} \hat{a}^2)/4\right] \end{aligned} \tag{34}$$

with

$$\Theta_\hbar = \coth(\beta \omega \hbar / 2)$$

These give again classical states in the sense of Definition 1, since at fixed $\beta, \{\rho_{\beta, \hbar}(\cdot) | \hbar \in (0, 1]\}$ converges uniformly on compacta as $\hbar \to 0$ to the function

$$\rho_\beta : z \in R^n \times \hat{R}^n \mapsto \exp\left[-\beta^{-1}(a^2 + \omega^{-2}\hat{a}^2)/2\right] \in C \tag{35}$$

It should be noticed that the Fourier transform of this function is the measure

$$d\mu_\beta(\hat{p}, q) = Z^{-1} \exp[-\beta H(\hat{p}, q)]\, d\hat{p}\, dq \tag{36a}$$

with

$$Z = \int d\hat{p}\, dq \exp[-\beta H(\hat{p}, q)] \tag{36b}$$

and

$$H(\hat{p}, q) = (p^2 + \omega^2 q^2)/2 \tag{36c}$$

which is the canonical equilibrium measure at natural temperature β for the classical harmonic oscillator.

All these examples suggest that classical states define distributions on $\hat{R}^n \times R^n$, and that one should thus expect the classical observables to be naturally defined, from the quantum observables, as functions on that space. This remark will be justified, and pursued further, in the next section.

4. CLASSICAL OBSERVABLES

We saw in Proposition 1 that every classical state $\{\varphi_\hbar \mid \hbar \in (0, 1]\}$ defines a continuous function

$$\varphi_0 : R^n \times \hat{R}^n \to C$$

of positive type and normalized to $\varphi_0(0) = 1$; this function in turn induces a linear map

$$\varphi_0 : f \in L^1_C(R^n \times \hat{R}^n) \mapsto \langle \varphi_0; f \rangle \in C$$

satisfying

$$\begin{array}{ll} \text{(i)} & \langle \varphi_0; f^* {}_* f \rangle \geqslant 0 \\ \text{(ii)} & \langle \varphi_0; f^* \rangle = \langle \varphi_0; f \rangle^* \\ \text{(iii)} & \| \varphi_0 \| = 1 \end{array} \tag{37}$$

These properties were shown to follow directly from the definition

$$\text{(iv)} \qquad \langle \varphi_\hbar; f\rangle \to \langle \varphi_0; f\rangle \qquad \text{as } \hbar \to 0 \tag{38}$$

Notice also the fact, easily extracted from the proof of the proposition, that

$$\text{(v)} \qquad \langle \varphi_\hbar; f *_\hbar g\rangle \to \langle \varphi_0; f * g\rangle \qquad \text{as } \hbar \to 0 \tag{39}$$

Hence our Definition 1 equips $L^1_C(R^n \times \hat{R}^n)$ with the following structure, which now becomes interpretable in quantum theory as well: it is an involutive Banach algebra under the usual convolution product, on which states are operationally defined and turn out to be positive linear forms normalized to 1. One could simply thus identify the classical observables as the elements of this algebra, in line with the usual view that an observable is empirically defined (see, e.g., Emch, 1972) by the values it takes on the possible states of the system under consideration. This, however, would not be yet the usual phase space formulation of classical mechanics. To derive the latter from the quantum premises adhered to so far in this paper, we need the following mathematical facts. Let $\mathfrak{S}$ be the set of all continuous functions from $R^n \times \hat{R}^n$ to C, which are of positive type; $\mathfrak{S}_1$ be the set $\{\varphi \in \mathfrak{S} \mid \varphi(0) \leqslant 1\}$; $\mathfrak{P}$ be the set of all φ in $\mathfrak{S}$ which are pure, i.e., for which $\varphi = \varphi_1 + \varphi_2$ with φ_1 and φ_2 in $\mathfrak{S}$ implies that φ_1 and φ_2 are proportional to φ; and let finally $\mathfrak{P}_1$ be the set $\{\varphi \in \mathfrak{P} \mid \varphi(0) = 1\}$. We call the elements of $\mathfrak{P}_1$ the classical pure states. In the flat case considered here, they are parametrized by $\hat{z} = (\hat{p}, q)$ running over $\hat{R}^n \times R^n$, the dual of $R^n \times \hat{R}^n$ considered as an Abelian group. The general Bochner theorem (see, e.g., Dixmier, 1969) asserts that to every φ in $\mathfrak{S}$ with $\varphi(0) = 1$ corresponds a positive measure, of norm 1, on $\mathfrak{S}$, say μ_φ, concentrated on $\mathfrak{P}_1$ and such that

$$\varphi(z) = \int_{\hat{R}^n \times R^n} d\mu_\varphi(\hat{z}) e_{\hat{z}}(z), \qquad \forall z \in R^n \times \hat{R}^n$$

with

$$e_{\hat{z}}(z) = \exp[-i(\hat{p}\cdot a + q \cdot \hat{a})] \tag{40}$$

Hence the Fourier transform of φ_0 has a natural meaning as the unique [since $L^1_C(R^n \times \hat{R}^n)$ is Abelian] decomposition of the state φ_0 into its pure-state components. We have thus

$$\langle \varphi_0; f\rangle = \int d\mu_\varphi(\hat{z}) \tilde{f}(\hat{z})$$

with

$$\tilde{f}: \hat{z} \in \hat{R}^n \times R^n \mapsto \int dz\, e_{\hat{z}}(z) f(z) \tag{41}$$

which expresses the classical observables as functions $\tilde{f}$ on the "phase space" $\mathfrak{P}_1$ of the classical pure states. A phenomenological justification of the Wigner–Moyal correspondence is thus obtained from first principles, and its full meaning is given by the following result.

Proposition 2. To every quantum observable

$$\hat{f} = \int_{R^n \times \hat{R}^n} dz\, f(z) \exp[-i(a \cdot P_\hbar + \hat{a} \cdot Q_\hbar)]$$

corresponds a classical observable, i.e., a function

$$\begin{aligned} \tilde{f}: \hat{z} = (\hat{p}, q) \in \hat{R}^n \times R^n \\ \mapsto \int_{R^n \times \hat{R}^n} dz\, f(z) \exp[-i(a \cdot \hat{p} + \hat{a} \cdot q)] \end{aligned}$$

such that

(i) For every classical state $\{\varphi_\hbar \mid \hbar \in (0,1]\}$

$$\lim_{\hbar \to 0} \langle \varphi_\hbar ; \hat{f} \rangle = \int_{\hat{R}^n \times R^n} d\mu_\varphi(\hat{z}) \tilde{f}(\hat{z})$$

where μ_φ is the Bochner measure corresponding to the continuous function φ_0 of positive type determined by $\{\varphi_\hbar \mid \hbar \in (0,1]\}$;

(ii) $$\lim_{\hbar \to 0} \langle \varphi_\hbar ; \hat{f} \cdot \hat{g} \rangle = \int_{\hat{R}^n \times R^n} d\mu_\varphi(\hat{z}) (\tilde{f} \cdot \tilde{g})(\hat{z})$$

where $\tilde{f} \cdot \tilde{g}$: $\hat{z} \in \hat{R}^n \times R^n \mapsto \tilde{f}(\hat{z}) \tilde{g}(\hat{z})$ for all f and g in $\mathfrak{K}(R^n \times \hat{R}^n)$;

(iii) $$\lim_{\hbar \to 0} \left\langle \varphi_\hbar ; [\hat{f}, \hat{g}] / i\hbar \right\rangle = \int_{\hat{R}^n \times R^n} d\mu_\varphi(\hat{z}) \{\tilde{f}, \tilde{g}\}(\hat{z})$$

where $\{\tilde{f}, \tilde{g}\}$: $\hat{z} \in \hat{R}^n \times R^n \mapsto$

$$\sum_{k=1}^{n} \left(\partial_{q_k} \tilde{f} \cdot \partial_{\hat{p}_k} \tilde{g} - \partial_{q_k} \tilde{g} \cdot \partial_{\hat{p}_k} \tilde{f}\right)(\hat{z})$$

for all f and g in $\mathfrak{K}(R^n \times \hat{R}^n)$.

Proof. (i) follows directly from Definition 1 and Bochner's theorem, as explained in the motivating remarks preceding the statement of the proposition. To obtain (ii) we notice that, by definition of the twisted convolution product: $\langle \varphi_\hbar; \hat{f} \cdot \hat{g} \rangle = \langle \varphi_\hbar; f *_\hbar g \rangle$; now a slight adaptation of the argument presented in the proof that φ_0 is of positive type (see Proposition 1) shows that $\langle \varphi_\hbar; f *_\hbar g \rangle$ tends, as $\hbar \to 0$, to $\langle \varphi_0; f * g \rangle$; since $\tilde{f} \cdot \tilde{g} = (f * g)^\sim$ this proves (ii). To prove (iii) we again use an argument akin to the proof of the positive type property of φ_0; specifically, for f and g continuous of compact support, we have by (9)

$$\{f, g\}_\hbar(z) = \int d\zeta\, f(\zeta) g(z - \zeta) \pi_\hbar(\zeta, z)$$

which we now compare [see (12)] with

$$\{f, g\}_0(z) = \int d\zeta\, f(\zeta) g(z - \zeta) \sigma(\zeta, z)$$

The proof now parallels that of Proposition 1, with $[\chi_\hbar(\zeta, z) - 1]$ replaced by

$$[\pi_\hbar(\zeta, z) - \sigma(\zeta, z)] = \left\{ \frac{\sin[\sigma(\zeta, z)\hbar/2]}{\sigma(\zeta, z)\hbar/2} - 1 \right\} \sigma(\zeta, z)$$

this leads to

$$\lim_{\hbar \to 0} \langle \varphi_\hbar; \{f, g\}_\hbar \rangle = \langle \varphi_0; \{f, g\}_0 \rangle$$

and the proof is completed by noticing that $\{f, g\}_0^\sim = \{\tilde{f}, \tilde{g}\}$. ■

5. CONCLUSIONS

The main result of this paper is the derivation of the phase space formalism of classical mechanics (Propositions 1 and 2) from the operator formalism of quantum mechanics (see Section 2) by a systematic exploitation of one single feature of the theory, namely the convergence of expectation values as $\hbar \to 0$ (Definition 1 in Section 3). Specifically, the results are (i) the natural introduction, in the quantum theory, of the phase space of classical mechanics as the space of classical pure states; (ii) the description of classical states as probability measures on phase space; (iii) the proof that the Jordan and Lie algebra structures of quantum mechanics (linearity, symmetric operator product, and quantum commutator) carry over to define unambiguously the Jordan and Lie algebra structures of classical mechanics (linearity, point-wise multiplication of functions on phase space, and Poisson bracket); (iv) the justification from first principles of the solution of the correspondence problem provided by the Wigner–Moyal correspondence rule.

As a commentary on the mathematical structures involved in the correspondence problem, one might want to remark that the classical limit appears in a very precise sense as a contraction of $*$-algebras. As in the case of the Lie group contraction governing the passage from relativistic to nonrelativistic physics, the limiting algebraic objects (at $c=\infty$, $\hbar=0$) are quite different from the original ones (at $c<\infty$, $\hbar>0$); the algebras $\{\hat{\mathfrak{A}}_\hbar \,|\, \hbar \in (0,1]\}$ are all made up of the same elements (operators in $\mathfrak{H}$), and their composition laws are similar in that for all of them the Jordan and the Lie algebra structures appear respectively as symmetrizations and antisymmetrizations of the same operator product. By contrast, as one reaches the classical theory (at $\hbar=0$) the Lie structure (Poisson bracket) of $\tilde{\mathfrak{A}}_0$ involves partial derivatives which are totally absent from its Jordan structure (point-wise product). The mathematical origin of this phenomenon appears clearly from the relation between the (twisted) convolution products in $\mathfrak{A}_\hbar$ and $\mathfrak{A}_0$ (see in particular the proof of Proposition 2): it can be traced back to the fact that relation (12), when coupled to the restriction that f and g belong to $\mathfrak{K}(R^n \times \hat{R}^n)$, gives a particularly simple, and explicit, meaning to the statement that the Moyal bracket tends to the Poisson bracket as $\hbar \to 0$.

Mathematically, the restriction placed in Proposition 2 (ii) and (iii), namely that the functions f and g belong to $\mathfrak{K}$ rather than merely to L^1, is a cheap way to ensure that their Fourier transforms be infinitely differentiable so that, in particular, the Poisson bracket may be iterated at will. To increase even further one's control over $\tilde{f}$ (and $\tilde{g}$), one might be prepared to restrict f to be also C^∞; $\tilde{f}$ extends then, as a Fourier–Laplace transform, to an entire analytic function. The function $\tilde{f}$ obtained in this way satisfies then very stringent growth conditions which do not even allow for simple polynomials, or even for functions of $\hat{p}$ (or q) alone. Extending the space to which $\tilde{f}$ belongs, so as to keep it entire analytic, but to allow for more liberal growth conditions at infinity, e.g.,

$$|\tilde{f}(\hat{z})| \leq C(1+|\hat{z}|)^N \exp(A \cdot |\mathrm{Im}\, \hat{z}|)$$

(see Hormander, 1969) would involve allowing f to be a distribution of compact support. At the same time $\hat{f}$ would in general become unbounded, and one would therefore have to restrict the choice of admissible quantum states. A detailed analysis of this type of question is possible (cf. e.g., Grossmann, Loupias, and Stein, 1968), but it would lead us too far away from the elementary level at which we wanted to present the evidence that classical mechanics can, in its fundamental aspects, be completely derived from quantum mechanics.

For certain unbounded operators $H_\hbar$, i.e., those which lead to well-behaved canonical equilibrium states, namely, such that $\mathrm{Tr}\exp(-\beta H_\hbar)$ is finite for all $\hbar$ and β, and such that the corresponding $\{\varphi_\hbar \mid \hbar \in (0,1]\}$ satisfy the condition of Definition 1 (see also Proposition 1), the properties of the corresponding classical observable might be accessible by the method used in discussing example (e) in Section 3. This actually brings us back very close to the original motivation for the Wigner–Moyal rule (Wigner, 1932). Also in this connection, one might notice that the results of Proposition 2 justify the use of the original quantum KMS condition to derive its classical analog, for instance in its static version (Gallavotti & Verboven, 1975; Aizenmann, Gallavotti, Goldstein, & Lebowitz, 1976):

$$\langle \varphi_0; \{\tilde{f}, \tilde{g}\}\rangle = \beta\langle \varphi_0; \tilde{g}\{\tilde{f}, \tilde{H}\}\rangle$$

We should finally comment on the fundamental role geometry played in the formulation discussed in this paper, and on the insight it provides into the general correspondence problem. We started with the action

$$(a, x) \in \mathfrak{G} \times M \mapsto a[x] \in M$$

of a locally compact Lie group $\mathfrak{G}$ on a manifold M, and assumed that $\mathfrak{G}$ acts transitively on M, so that M is a homogeneous manifold, and inherits canonically the measure dx from the Haar measure on $\mathfrak{G}$. We then based our quantum mechanical description on the irreducible system of imprimitivity resulting from the action of $\mathfrak{G}$ on M; it is indeed well known (Mackey, 1963a) that this concept can be formulated in the full generality of the structure just outlined. Then, to avoid redundancies, we used the fact that there exists in $\mathfrak{G}$ a subgroup G which acts transitively *and* freely on M. This feature is encountered again in significantly more general circumstances, including nonflat manifolds. For instance, if M were chosen to be the Poincaré half-plane

$$M = \{w = x + iy \mid x \in R,\, y \in R^+\}$$

endowed with the metric $g(dx, dy) = (dx^2 + dy^2)/y^2$; and $\mathfrak{G}$ were

$$SL(2, R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in R, ad - bc = 1 \right\}$$

with the action of $\mathfrak{G}$ on M given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \in M \mapsto \frac{aw + b}{cw + d} \in M$$

Upon taking $w_0 = i$ for origin, we introduce its stabilizer

$$K = \{ a \in \mathfrak{G} | a[i] = i \} = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} | \varphi \in [0, 2\pi) \right\}$$

M is then recovered as the homogeneous manifold $\mathfrak{G}/K$. G can then be taken to be

$$G = \left\{ \begin{pmatrix} e^{-t} & s \\ 0 & e^{t} \end{pmatrix} | s, t \in R \right\}$$

We further remark that G contains two closed subgroups

$$N = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} | s \in R \right\}; \qquad A = \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} | t \in R \right\}$$

such that every $a \in G$ can be written uniquely as $a = \nu \cdot \alpha$ with $\nu \in N$ and $\alpha \in A$. In fact N, A, and K are characterized uniquely (up to conjugacy) in $\mathfrak{G}$ by properties (i)–(v) below (Iwasawa decomposition): (i) N, A, and K are closed subgroups of $\mathfrak{G}$; (ii) N is nilpotent; (iii) A is a real vector group which normalizes N; (iv) K is compact, and maximally so in $\mathfrak{G}$; (v) every g in $\mathfrak{G}$ can be written uniquely as $g = \nu \cdot \alpha \cdot k$ with $\nu \in N$, $\alpha \in A$, and $k \in K$. However, $\mathfrak{G}$ differs from the Euclidean group in three aspects: (vi) $\mathfrak{G}$ is not a semidirect product of G by K (as opposed to E^n which is a semidirect product of the translation group R^n by the rotation group O^n); (vii) G is not Abelian: all that is left from the Abelianness of the translation group is property (iii) above; (viii) whereas N, A, and K are unimodular [for $\mathfrak{G} = SL(2, R)$], G is not. We also note that the same structural properties obtain for $\mathfrak{G} = SL(2, C)$ with now M diffeomorphic and isometric to the three-dimensional (instead of two-dimensional) mass hyperboloid in Minkowski $3+1$ (instead of $2+1$) space. Here again (Mackey, 1968, pp.

126–127; Mackey, 1975) the essential features, except for flatness, of the usual Euclidean case are maintained; namely: $\mathfrak{G}$ is a semisimple Lie group with finite center, and no compact component; K is a maximally compact subgroup of $\mathfrak{G}$, unique up to conjugacy; and $M=\mathfrak{G}/K$, on which G acts transitively and freely, is a homogeneous Riemann manifold [for the conceptual framework behind these properties, see, e.g., Godement (1952), Mackey (1963b), and the Appendix in Mackey (1976)]. These structures are thus the first natural candidates to which the considerations of the main body of this paper should be extended.

We thus briefly comment on the geometrical and physical meaning of $\hbar$ in this scheme. In the flat case, $\hbar \in R^{+}$ appeared in formulas (16) and (17) as a parametrization of all extensions of $R^n \times \hat{R}^n$ by Ω, which are invariant under the natural action of O^n on $R^n \times \hat{R}^n$. More generally, this can be understood as follows. For every $x_0 \in M$, the map

$$i_{x_0}: a \in G \mapsto a[x_0] \in M$$

is surjective (since G acts transitively on M) and injective (since G acts freely on M); it thus establishes a vector space isomorphism between the tangent space $(TM)_{x_0}$ of M at x_0 and the Lie algebra $\mathfrak{L}$ of G. Tracing now where $\hbar$ entered the theory, namely, back in formula (2), we see that a choice of $\hbar>0$ corresponds exactly to a choice of a unit of speed for the geodesics of M issued from x_0: it thus fixes a scale in $\mathfrak{L}$, and can therefore be interpreted as a choice of a multiplicative factor in the Riemann metric of M [or equivalently, as a choice of a mass unit (Mackey, 1963a) in the kinetic energy, i.e., in the Hamiltonian of the geodesic flow on M]. With M, G, and $\hbar$ generalized as just indicated, one can define the imprimitivity system (1), the irreducible operator-algebra $\hat{\mathfrak{A}}_{\hbar}$ on $L^2(M, dx)$, and the function-algebra $\mathfrak{A}_{\hbar}$. This can even be done in a coordinate-free manner, although specific coordinates will ultimately enter in the identification of the momentum operator $P_{\hbar}^{(k)}(k=1,2,\ldots,n)$ conjugate to the position operator $Q_{\hbar}^{(k)}$. One should moreover be aware that two slight complications will occur when M is allowed not to be flat, as illustrated above by the example of the Poincaré half-plane. First, it will not be possible in general to choose n one-parameter subgroups of G in such a manner that the image, through i_{x_0}, of each one of them be a geodesic in M; since, however, the limit $\hbar \to 0$ does concentrate on the Lie algebra $\mathfrak{L}$ of G, this will be of little consequence. Second, G will in general not be Abelian, so that the $P_{\hbar}^{(k)}$'s (in contrast to the $Q_{\hbar}^{(k)}$'s) will not necessarily commute among themselves; this will evidently show in the nonvanishing of the Poisson brackets between the generators of G, the latter appearing now in the classical theory as functions defined on the cotangent bundle $T*M$ of M.

Modulo these mathematical and epistemological precautions, the theory will thus proceed along the path described in the main body of this paper, provided that M is a simply connected homogeneous Riemann manifold, on which a Lie group G acts transitively and freely. Further adaptations, to which we intend to come back more specifically in a sequel to the present paper, become necessary when M is allowed, for instance, to be a multiply connected manifold of strictly negative curvature, compact and without boundaries, such as those which support classical Anosov flows, e.g., where $M = \Gamma \backslash \mathfrak{G} / K$ with Γ a discrete cocompact subgroup of $\mathfrak{G}$.

In view of the status of the no-go theorems alluded to in Section 1, and already in force in R^n, it was thought proper to first present the dequantization program in the form of a detailed, elementary, but complete solution to the correspondence problem in R^n. In doing so, we settled affirmatively a conjecture proposed in Mackey (1963a, pp. 103–104), namely, that *it is possible to start with a theory which makes no assumptions beyond the fundamental principles of quantum mechanics, and to derive rigorously from these premises the complete structure of the corresponding classical mechanics.*

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